Persistence in one-dimensional Ising models with parallel dynamics

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We study persistence in one-dimensional ferromagnetic and antiferromagnetic nearest-neighbor Ising models with parallel dynamics. The probability $P(t)$ that a given spin has not flipped up to time t, when the system evolves from an initial random configuration, decays as $P(t) \sim 1/t^{\theta_p}$ with $\theta_p \simeq 0.75$ numerically. A mapping to the dynamics of two decoupled $A+A \rightarrow 0$ models yields $\theta_p = 3/4$ exactly. A finite size scaling analysis clarifies the nature of dynamical scaling in the distribution of persistent sites obtained under this dynamics.

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Recent work on the phenomenon of ''persistence'' has revealed an unusual (and in many respects unexpected) way in which an interacting many-body system, evolving in time, retains memory of its initial state [1]. Consider a one-dimensional ferromagnetic Ising model with nearest-neighbor interactions, quenched from an initially random (infinite temperature) configuration and allowed to relax to its global minimum energy (zero temperature) configuration, a state with either all spins up or all spins down. Suppose the dynamics is serial, with an attempt to update a single spin being performed at each time step. Fix one spin and ask: What is the probability that this spin has *not* flipped up to time t? This quantity, the persistence probability, was first found numerically to decay as

$$P(t) \sim \frac{1}{t^{\theta_s}}, \tag{1}$$

with θ_s a nontrivial exponent. (The subscript refers to serial dynamics.) The numerical results suggested $\theta_s \sim 0.37$ [2]; a later analytical investigation derived $\theta_s = 3/8$ exactly [3]. Later studies have established rigorously that the persistence exponent is a different exponent characterizing the dynamics; it cannot be related to either the static exponents ν and η or the dynamical exponent z [1].

This paper discusses the problem of persistence in the context of one-dimensional Ising models with nearest-neighbor interactions evolving under *parallel* dynamics. We consider both ferromagnetic and antiferromagnetic interactions, where the Hamiltonian has the form

$$H = -J \sum_i \sigma_i \sigma_{i+1}. \tag{2}$$

Without loss of generality, we take $J=1$ for ferromagnetic interactions and $J=-1$ for the antiferromagnetic case. Each spin can take values $+1$ (up) or -1 (down). The zero temperature dynamics evolves a configuration $\{\sigma(t)\}$ at time t to a configuration $\{\sigma(t+1)\}$ at time $t+1$ through the following simple rule: For ferromagnetic interactions, each spin at time $t+1$ assumes the value of one of its neighboring spins at time t, chosen from right or left with equal probability. For antiferromagnetic interactions, the above rule is modified in the following way: At time $t+1$, assign the value of each spin to the *negative* of the value of one of its neighboring spins, chosen from right or left with equal probability. Each such step in time constitutes a single Monte Carlo step. The parallel nature of the dynamics follows from the fact that all spins are updated together.

We have simulated parallel dynamics using the above rules on Ising systems of linear size $L = 10^2 - 10^6$ sites and for times $t \leqslant 10^5$, applying periodic boundary conditions. We average over a fairly large number of initial conditions, typically $10^2 - 10^3$ for the smaller lattices ($L < 10^4$), starting from configurations in which each spin is independently assigned a value 1 or -1 with equal probability. We compute the standard persistence probability $P(L,t)$, defined as the probability that the spin at a given site in a system of size L has not flipped up to time t, averaged over all sites and over an ensemble of initial conditions. For $L \rightarrow \infty$ (in practice for $t \ll L^z$), $P(L,t) \rightarrow P(t)$.

Figure 1 shows the persistence probability $P(t)$ for a ferromagnetic Ising system of linear size $L = 10^6$, evolving under parallel dynamics. $P(t)$ exhibits a power law tail with an exponent $\theta_p \sim 0.75$. The behavior is identical for antiferromagnetic interactions. For comparison, the corresponding plot for serial dynamics is shown on the same figure; the exponent, as advertised, is $\theta_s = 3/8$ to within numerical resolution. The exponent $\theta_s = 3/8$ is also obtained for antiferromagnetic interactions under serial dynamics. It is thus natural to guess that $\theta_p = 3/4 = 2\theta_s$, and that θ_p as well as θ_s remain unaltered when ferromagnetic interactions are replaced by antiferromagnetic ones.

Coarsening in ferromagnetic Ising models with serial dynamics occurs through the motion and annihilation of do-

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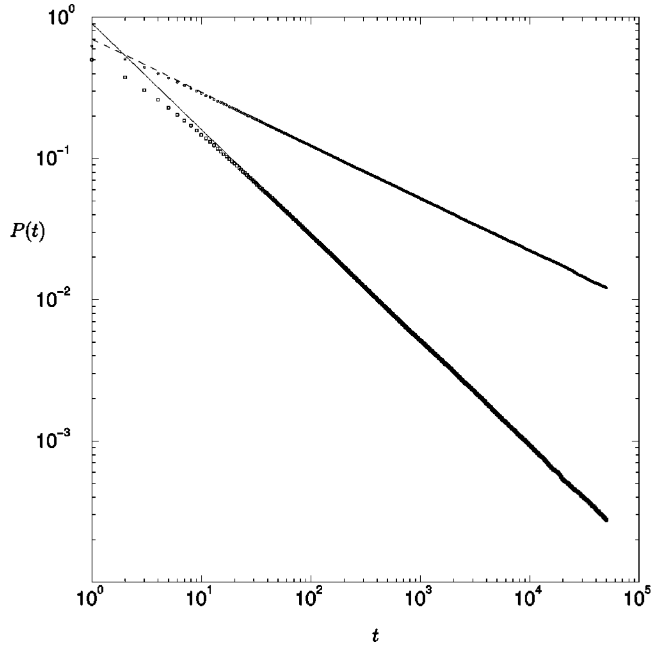

FIG. 1. The persistence probability $P(t)$ in a one-dimensional Ising model plotted against time t in a logarithmic scale. The lower curve (squares) is for parallel dynamics while the upper one (circles) is for serial dynamics. The solid and the dashed lines fitted to these curves have slopes 0.75 and 0.375, respectively. The slopes remain the same if ferromagnetic interactions are replaced by antiferromagnetic interactions.

main walls separating spins with different orientations. Spins located at domain walls have no preference for either orientation. A given spin flips if it is crossed by a domain wall. Thus, persistence in this context is equivalent to the probability that a specified site has not been crossed by a domain wall up to time t. If the domain walls in this problem are interpreted as particles of type A, the following simple reaction-diffusion scheme describes the motion and annihilation of domain walls: $A+A\rightarrow 0$, with particles diffusing at each time step and annihilating on contact [4].

The problem of persistence with serial dynamics then translates simply into the following: Given a chosen site at time $t=0$ and an initial configuration of A particles, corresponding to domain walls in the initial configuration, what is the probability that an A particle has not crossed that site up to time t? Such a redefinition of the problem recasts the question in terms of the essential ingredients of the dynamics, the motion and interaction of domain walls. It is natural to look for the analog of such domain walls in the case of parallel dynamics to understand the conjectured relationship $\theta_p=3/4=2\theta_s$.

Consider first the ferromagnetic case and divide configurations of spins into the following categories: unstable spins, implying that they will definitely flip in the next time step, stable spins, implying that they will not flip in the next time step, and ‘‘zero-field’’ spins, which may or may not flip, with either possibility occurring with probability 1/2. An unstable spin σ_i at site i has both neighbors in the state $-\sigma_i$, while stable spins point in the same direction as their neighbors. Zero-field sites, on the other hand, have one neighbor pointing up while the other points down. As a consequence, flipping the spin at a zero-field site costs no energy.

These zero-field sites are the analogs, for parallel dynamics, of domain walls in the ferromagnetic Ising model with serial dynamics, in a sense that we make precise below. Stable sites belong to domains that are either all up or all down. A spin in the bulk of such a domain can only increase its energy if it flips; it is thus not updated through the zero-temperature dynamics. Unstable sites, for the case of ferromagnetic interactions, are associated with an antiferromagnetic arrangement of spins of the form ...1010101010..., where the notation ‘‘1’’ indicates an up spin and ‘‘0’’ indicates a down spin. Note that all sites interior to such a region *will* flip in the next time step. Within a region of unstable sites, spin histories follow a two-cycle; each spin flips once in each time step. Sites within such regions cannot contribute to persistence, for they cannot be persistent beyond a single time step. It is obvious that persistence of spin configurations at late times can only be associated with spins that lie deep within stable regions.

It is useful for the ensuing discussion to divide the one-dimensional lattice into two interpenetrating sublattices A and B. For example, we may take all even-numbered sites as forming the A sublattice and odd-numbered sites as constituting the B sublattice. The state of sublattice A (B) at any time t is determined by the state of sublattice B (A) at $t-1$. The initial states of sublattices A and B are uncorrelated with each other.

We will argue below that the persistence of a site i on a given sublattice at times $t=1,3,5,\dots,(2m+1)$ is determined by the configuration of zero-field sites on the other sublattice, while at times $t=2,4,6,\dots,2m$ it is determined by the configuration of zero-field sites on its own sublattice. With this in mind, it is useful to distinguish zero-field sites on sublattice A from those on sublattice B. Call every zero-field site on the A sublattice an A particle and every zero-field site on the B sublattice, a B particle, at time $t=0$. Figure 2 illustrates a sequence of configurations at succeeding instants of time through which an initial state evolves. The letters A and B indicate the positions of zero-field sites on the A and B sublattices.

Inspection of Fig. 2 leads to the following understanding. Zero-field sites are located at (i) the interface between two stable regions, in which case this interface is composed of two adjacent zero-field sites—this is the case for the model with serial dynamics in which case A particles represent the bound pair of zero-field sites, (ii) the interface between a stable and an unstable region, in which case there is a single zero-field site, and (iii) the interface between two unstable regions, in which case there is again a pair of adjacent zero-field sites. The motion of zero-field sites corresponds to the expansion/contraction of one or the other domain it separates. Zero-field sites can also annihilate, leading to the coalescence and shrinkage of domains, as the system coarsens under the dynamics.

It is thus obvious that the nontrivial dynamics associated with the persistence phenomenon can be associated only with the zero-field sites, for such sites constitute the bound-

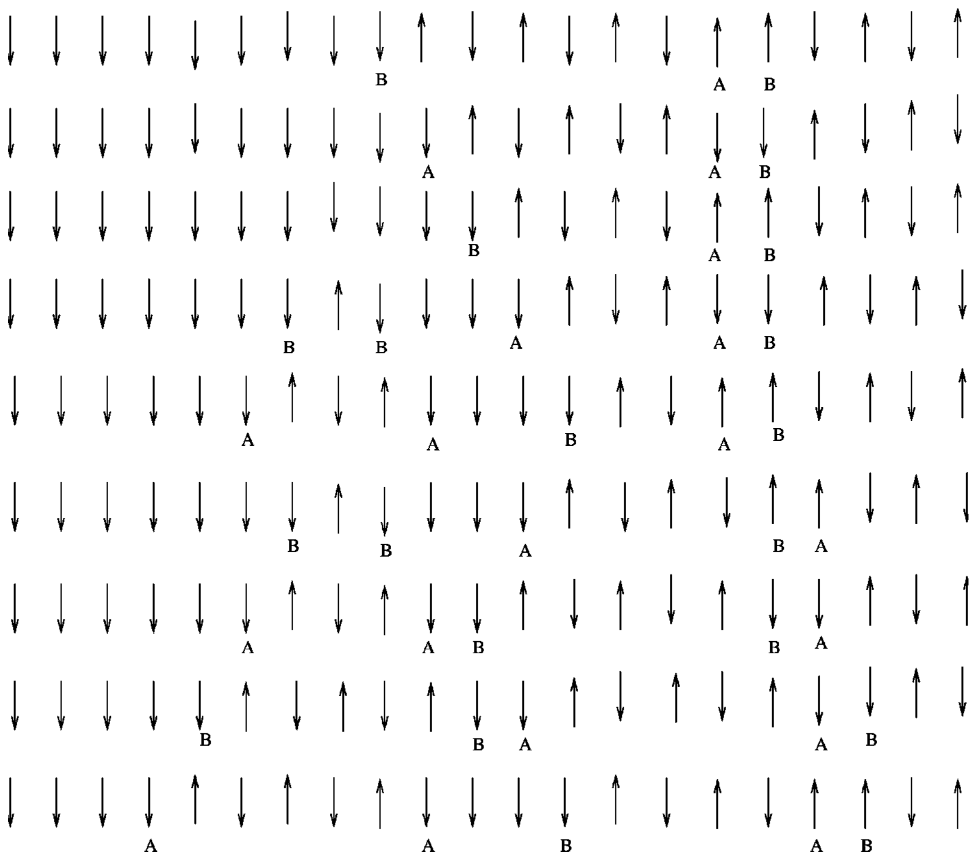

FIG. 2. Time evolution of the zero-field sites on *A* and *B* sublattices: the configuration at the earliest time corresponds to the bottom row and each successive row is a later time step. Note that *A* and *B* particles as defined at any given time step do not react with each other. Only particles of the same kind, i.e., *A* or *B*, can annihilate.

aries between unstable and stable regions. It is also obvious that sites which are persistent until time t can be associated only with regions that are stable up to time t, i.e., regions that are stable at $t=0$ but that are not crossed by a zero-field site up to time t. The problem of persistence in the case of parallel dynamics can thus be discussed precisely as in the serial case, with the dynamics of the simple domain walls in the serial case being replaced by a somewhat more complex dynamics in the parallel case, which we deal with below.

We now establish the following crucial ingredient of this model which enables the exact calculation of θ_p, using the known result for θ_s and the mapping onto the equivalent reaction-diffusion problem: Only particles of the same type (A or B) can annihilate each other. They are transparent to particles of the other type. Thus, the problem of the motion of zero-field sites in this model reduces to the problem of two *decoupled* $A+A\rightarrow 0$ and $B+B\rightarrow 0$ models, with the initial number of A's and B's being set by the initial conditions. This is clearly evident from Fig. 2 if we look at the configurations at even or odd time steps and at even or odd lattice sites. Our use of periodic boundary conditions implies that $N(A)$ and $N(B)$ are always even, where $N(A)$ and $N(B)$ represent the number of particles of type A and B at any time.

The *single* conservation law of the $A+A\rightarrow 0$ model, the conservation of particle number $N(A)$ mod 2, is replaced by two conservation laws in this model: $\mathrm{mod}[N(A),2]=0$ and $\mathrm{mod}[N(B),2]=0$, where $N(A)$ and $N(B)$ are the numbers of A and B particles in any configuration of the model. It is useful to note the following: while the properties conventionally computed for the reaction-diffusion scheme $A+A\rightarrow 0$ refer to sequential dynamics, these results generalize trivially to parallel dynamics. This is a consequence of the fact that particle moves are independent of each other in this model. Only the reaction itself, in which two A particles on the same site annihilate each other, has different interpretations in parallel and serial dynamics, but this ambiguity is easily removed through a simple redefinition of the time and length scales.

For the $A+A\rightarrow 0$ model, the density of A particles decreases as [5]

$$N(A)\sim 1/t^{1/2}. \tag{3}$$

In the context of the Ising model with parallel dynamics, this result implies that $N(A)$ and $N(B)$ as defined above and at times $t, t+2, t+4, \ldots$ are conserved modulo 2 and decay *separately* as $1/t^{1/2}$, with a prefactor that depends on the initial concentration. This result accords with results obtained for this model by Privman [4]; we have checked this numerically as well. Also note the following: If we consider only the persistence of sites on a given sublattice, at only even or odd time steps, i.e., $t=0,2,4,\ldots,2m$ or $t=1,3,5,\ldots,2m+1$, this will decay as $P(t)\sim 1/t^{3/8}$, reflecting the fact that this dynamics maps *exactly* onto the $A+A\rightarrow 0$ dynamics. This is consistent with our numerics.

We now use these results to argue the following. Consider, for concreteness, the persistence of a spin on the A sublattice at some time t after the quench from infinite temperature. The persistence probability is then simply the probability that the site in question has not been crossed by an A

particle up to time t *or* a B particle up to time $t-1$. Since these probabilities are independent (the dynamics of A and B particles decouples), this joint probability is simply the product of the independent probabilities that the site is persistent with respect to the motion of both A and B particles, implying

$$P(t) \sim \frac{1}{t^{3/8}} \times \frac{1}{(t-1)^{3/8}} \sim \frac{1}{t^{3/4}}, \tag{4}$$

yielding the persistence exponent for parallel dynamics $\theta_{par} = 3/4$ exactly, consistent with the numerical data.

To test the validity of the mapping onto the two noninteracting species of particles (A and B) outlined above, we have simulated the associated reaction-diffusion model independently and computed, numerically, the analog of the persistence probability for the Ising case. This is done by computing the probability that a given site is crossed by a particle of neither type up to time t; the exponent obtained numerically tallies precisely with our result above.

How do these results generalize to the antiferromagnetic Ising model with parallel dynamics? These results are unaltered as a consequence of the following simple mapping of configurations: Replacing all spins on one sublattice, say the A sublattice, through $\{\sigma_A\} \rightarrow -\{\sigma_A\}$ changes the sign of the exchange interaction J, mapping the problem with the new variables onto the ferromagnetic problem. This gauge symmetry relates configurations pairwise; every update for the ferromagnetic case is an allowed update for the antiferromagnet with the same weight. Thus none of the conclusions here are altered and the persistence exponent is independent of the *sign* of the exchange interaction J.

Recently, there has been considerable interest in the spatial scaling properties of persistence [6]. Numerical work on the one-dimensional $A+A\rightarrow 0$ model, which describes the Ising model with *serial* dynamics, shows the existence of a nontrivial fractal structure in the spatial distribution of persistent sites at long times. These results can be recast in terms of a dynamical scaling form for $P(L,t)$ [7]:

$$P(L,t) = L^{-z\theta_s} f(t/L^z), \tag{5}$$

where z is the dynamic exponent and $f(x) \sim x^{-\theta_s}$ for $x \ll 1$ while it is constant for large x. One consequence of this form is that persistent sites at long times and for length scales $l \ll t^{1/z}$ constitute a fractal with fractal dimension $d_f = d - z\theta_s = 0.25$. We have used $z=2$, valid for $A+A\rightarrow 0$ dynamics.

Does such structure exist for the parallel version of Ising persistence? Our data for $P(L,t)$ are consistent with Eq. (5), with θ_s replaced by θ_p and $z \simeq 2$, as shown in the scaling plot of Fig. 3. This illustrates the validity of the dynamical scaling *ansatz* for persistence under parallel dynamics. (Data collapse here is, however, inferior in comparison to the serial case.) Using the result of the previous paragraph, this would indicate a fractal dimension of -0.5, *a priori* an unphysical result. This result can be attributed to the fact that we are looking at the persistence of a site under two *independent*

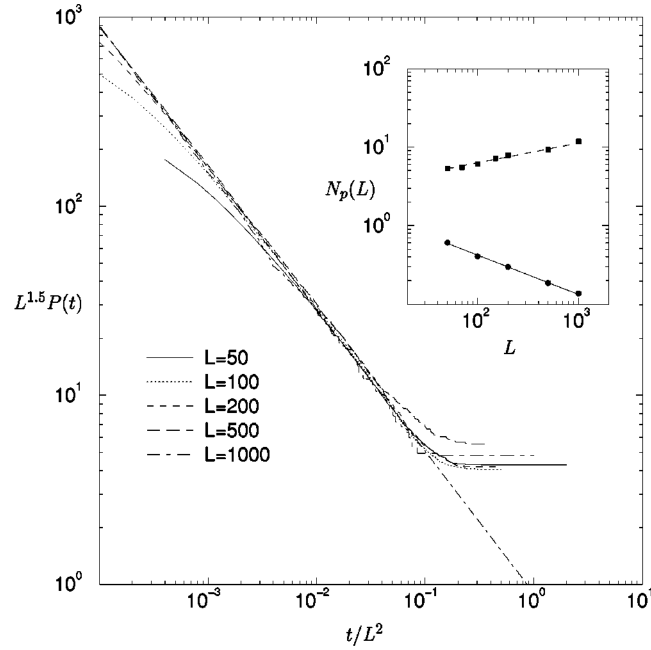

FIG. 3. Plot of $P(L,t)L^{z\theta_p}$ vs t/L^z, where $P(L,t)$ is the configuration averaged density of persistent sites in a system of size L at time t, $z=2$, and $\theta_p = 3/4$, illustrating the validity of the dynamical scaling ansatz. The inset shows the total number of persistent sites $N_p(L) = LP(L, t=\infty)$ left in the system as $t\rightarrow\infty$, plotted against the system size L on a logarithmic scale, for both parallel (circles) and sequential (squares) dynamics. The straight lines fitted to these points have slopes -0.5 and 0.25, respectively (see text for discussion).

processes. Consider the set of sites persistent with respect to the motion of A and B particles separately; each will form a fractal with the same fractal dimension d_f. The intersection of these two fractals represents those sites persistent with respect to the motion of both A and B particles. Assuming that A and B particles are initially uncorrelated, the dimension of the intersection set is then $2d_f - d = -0.5$, as above. We conclude that persistent sites in the parallel dynamics version of the Ising model do *not* exhibit spatial scaling of the type seen in the serial version of the model, a result corroborated by the work of Bray and O'Donoghue [8] and Manoj and Ray [7].

One implication of this result is that both the average *number* and average density of persistent sites in a system of size L should *decay* with L for parallel dynamics, in contrast to the serial case. We have verified this numerically; see the inset to Fig. 3. (In contrast, the mass of a truly fractal object *increases* with scale while its density decreases.) Thus, a large system has no persistent sites, at sufficiently long times, for most initial conditions. The power law tail of $P(L,t)$ is then associated with an ever smaller fraction, as L increases, of initial states, whose associated persistent sites survive for longer and longer times.

In conclusion, we have studied persistence in one-dimensional Ising models with parallel dynamics. We have obtained the persistence exponent $\theta_p = 3/4$ exactly and studied the spatiotemporal correlations of persistent sites. These results relied on an exact mapping to the dynamics of two *decoupled* $A + A \rightarrow 0$ models. It would be interesting to see if similar arguments exist and are useful in the discussion of persistence with parallel dynamics in other models.

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